Free Boundary Problems as Hamilton-Jacobi-Bellman equations

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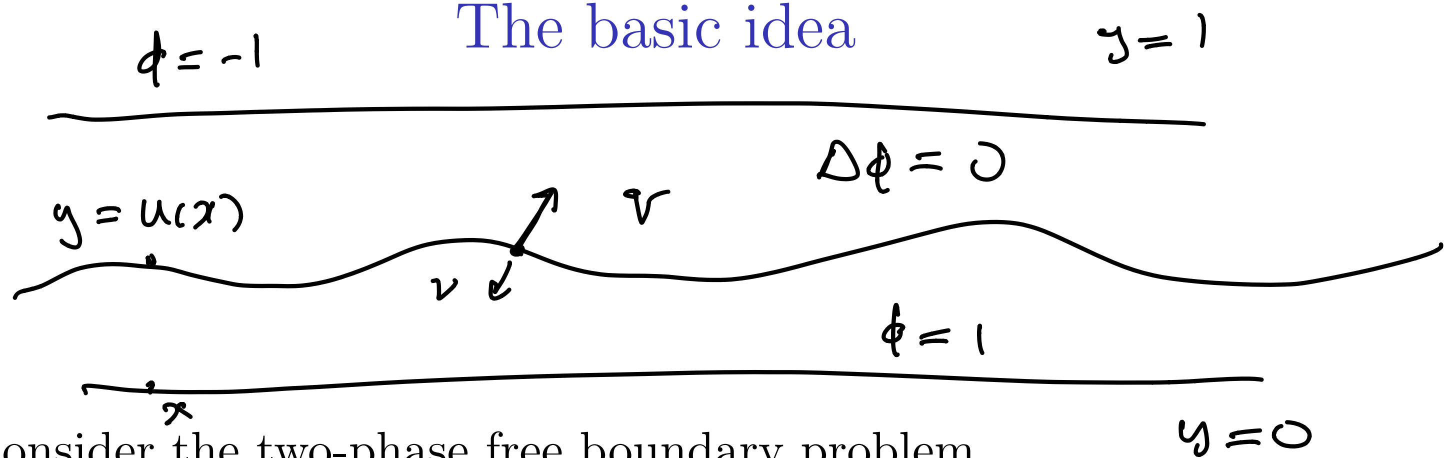
Online Analysis and PDE Seminar
March 2021

Partially based on past and ongoing works with Russell Schwab and Héctor Chang-Lara.

1.The basic idea

The basic idea



Consider the two-phase free boundary problem

$F_1(D^2\phi) = 0$

$F_2(D^2\phi) = 0$

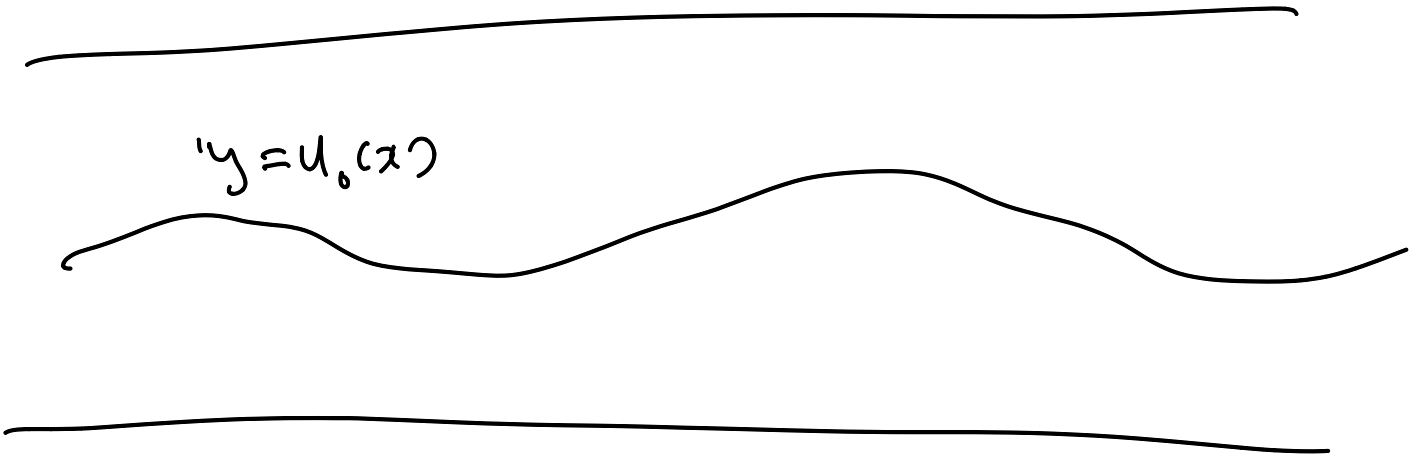
$$\Delta\phi = 0 \text{ in } \{\phi > 0\}$$
$$\Delta\phi = 0 \text{ in } \{\phi < 0\}$$
$$V = G(\partial_\nu^+\phi, \partial_\nu^-\phi) \text{ on } \partial\{\phi > 0\}$$

$\partial_\nu^+\phi - \partial_\nu^-\phi$

Posed on the strip $\mathbb{R}^d \times [0, L] = \{(x, y) \mid 0 \leq y \leq L\}$ with

$$\phi \equiv 1 \text{ on } \{y = 0\}, \ \phi \equiv -1 \text{ on } \{y = L\}.$$

The basic idea



Theorem (with Chang-Lara and Schwab, 2019)
Consider an initial data ϕ_0 where

$$\{\phi_0 = 0\} = \{ \text{ graph of } u_0 \}$$

u_0 a continuous function. There is a unique weak solution starting from ϕ_0 and defined for all $t > 0$ whose interface is the graph of a continuous function $u(x, t)$.

The basic idea

interface

This theorem will result from the observation that $u(x,t)$ solves

$$\partial_t u = I(u)$$

where I is a **degenerate elliptic** operator

The basic idea

What does this mean?
The free boundary problem is equivalent to $\partial_t u = I(u)$, an equation amenable to treatment by non-divergence methods (i.e. comparison/barrier arguments and Krylov-Safonov theory)

Think for instance of equations of the form

PME

$$\partial_t u = u\Delta u + |\nabla u|^2$$

$$\partial_t u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad \text{p- Laplace}$$

$$\partial_t u = \max_\alpha\{\mathrm{tr}(A_\alpha D^2 u)\} \quad \text{Bellman eqn.}$$

but more integro-differential!

The basic idea

…"more integro-differential" would be for instance

$$\partial_t u = \Delta^{\frac{\sigma}{2}} u,\ \sigma \in [0,2] \quad \leftarrow \quad \sigma = 1$$

$$\partial_t u = \max_{\alpha} \left\{ \int_{\mathbb{R}^d} \delta_h u(x) K_\alpha(x,h)\, dh \right\} \quad \leftarrow \tilde{\tilde{I}}$$

$$\partial_t u = \int_{\mathbb{R}^d} F(u(x+h) - u(x), h)\, dh \quad \leftarrow$$

Here $K_\alpha \geq 0$ for all α, F is increasing with its first argument

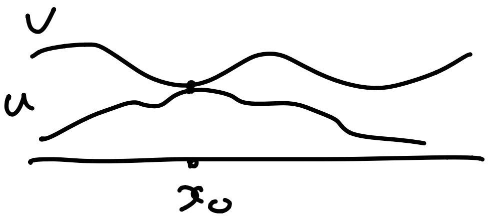
These are instances of **Hamilton-Jacobi-Bellman equations**

Isaacs

The basic idea

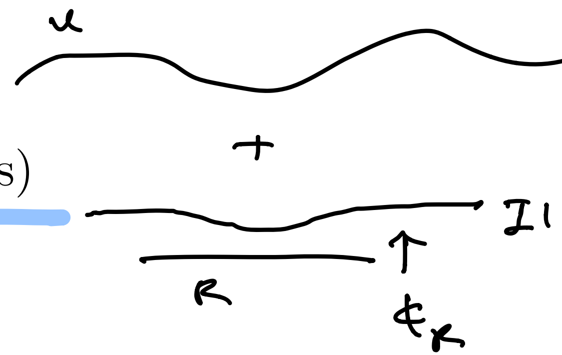
"what you need for the max pple to work"

(Global Comparison Property)
If $u \le v$ for all x and $u = v$ at x_0, then



$$I(u) \le I(v)$$

x_0 x_0



(Perturbation under smooth bump functions)
Let u lie in some fixed compact set.
Then, given $\varepsilon > 0$ there is $R > 0$ such that

$$I(u + C + h\phi_R, x) < I(u, x) + \varepsilon, \; \forall\, C, h > 0$$

Here, $\phi_R(x) = \phi(x/R)$ and $\phi(x) = |x|^2/(1 + |x|^2)$

The basic idea

Under these circumstances, the **Comparison Principle** holds.

Let $u, v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ be bounded, continuous, and

$$\text{Subsolution} \rightarrow \partial_t u \leq I(u) \text{ and } \partial_t v \geq I(v) \leftarrow \text{supersolution}$$

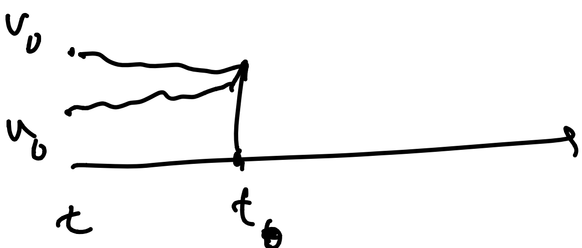
If $u(x, 0) \leq v(x, 0)$ for all x, then $u(x, t) \leq v(x, t)$ for all x, $t > 0$.

The basic idea

(Here is a **romantic** proof of how this goes)

$u_0 \le v_0$, $u > v$ at a later time

$\exists\ (x_0, t_0)$ s.t.



$u(x, t_0) \le v(x, t_0) \quad \forall x$

$u = v$ at x_0, t_0

$I(u, x_0) \ge \partial_t u(x_0, t_0) \ge \partial_t v(x_0, t_0)$

$\ge I(v, x_0)$

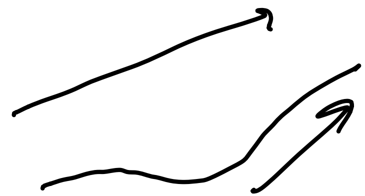
$\ge I(u, x_0)$

The basic idea

(Here is a **romantic** proof of how this goes)

If $u - v > 0$ at some $t > 0$, one may choose $C, h > 0$ such that

$$b(x,t) := C + h\phi_R(x) + \varepsilon t$$



touches $u - v$ from above at some (x_0, t_0).

$\Rightarrow$ u is touched from above

by $v + b$

The basic idea

Equivalently, $v + b$ touches u from above at (x_0, t_0). Then,

$$\partial_t u \geq \partial_t (v + b) \text{ and } I(u) \leq I(v + b) \text{ at } (x_0, t_0).$$

Property #2 $\quad \underline{I(v + c + h\phi_r) < I(v) + \varepsilon h}$

It follows that $\partial_t(v + b) \geq I(v + b)$ at (x_0, t_0). However!

$$\partial_t(v + b) = \partial_t v + \varepsilon$$
$$I(v + b) < I(v) + \varepsilon$$

In contradiction with $\underline{\partial_t v \leq I(v)}$ eveywhere. $\square$

$\Rightarrow \quad u \leq v \quad \forall\, t > 0$

2.Examples of Interfacial Darcy Flows

Interfacial Darcy Flows

The term *Interfacial Darcy Flows* was introduced by Ambrose to describe a rich family of models combining these two features:

1. An incompressible flow $\boldsymbol{v}$ satisfying Darcy's law

$$\boldsymbol{v} = -K\nabla\phi$$

2. An interface evolving along with the flow, meaning

$$\begin{gathered}\operatorname{div}(K\nabla\phi) = 0 \text{ away from } \Gamma \\ + \text{ some conditions for } \phi, V \text{ on } \Gamma\end{gathered}$$

Interfacial Darcy Flows

In these flows the interface velocity V is determined by ϕ, and ϕ is determined by Γ.

Naturally, this means Γ evolves according to a nonlocal process. This allows for their treatment as an abstract evolution equation for Γ. Several well-posedness theories, local and global, have been developed through this philosophy

Interfacial Darcy Flows

Hele-Shaw cell (2D)

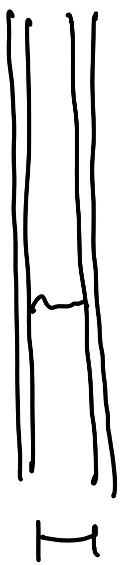
pressure

density

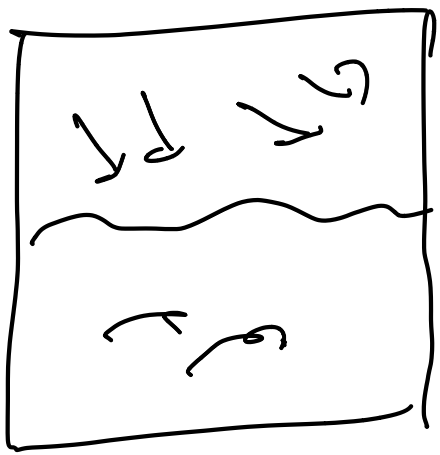
gravity

$$\left[\ \boldsymbol{v}(x) = -\frac{b^2}{12\mu}\nabla(p(x) + \gamma g x_{d+1})\right.$$

visc.

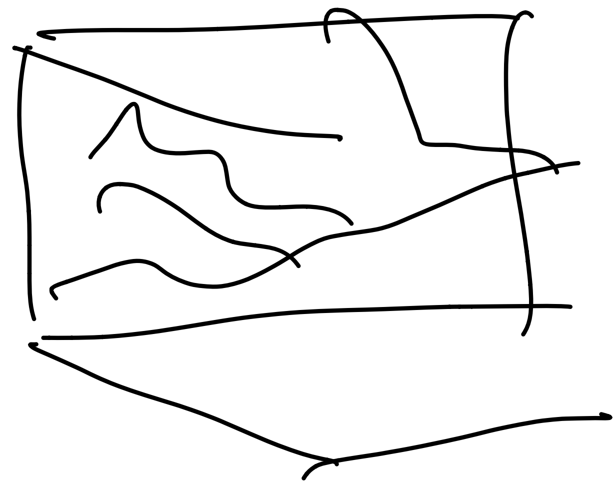


Front view

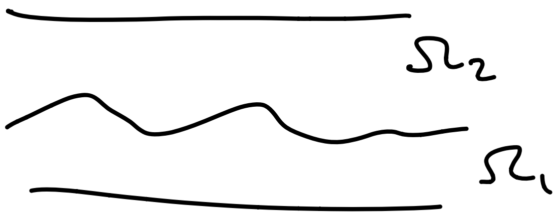


Porous media flow (3D)

$$\boldsymbol{v}(x) = -\frac{\kappa}{\mu}\nabla(p(x) + \gamma g x_{d+1})$$



Interfacial Darcy Flows



Example 1: The Muskat Problem for two immiscible fluids

$\mu_1 \neq \mu_2$

$\gamma_1 \neq \gamma_2$

$$\begin{cases} \operatorname{div}(\boldsymbol{v}) = 0 \\ \boldsymbol{v}(x) = -\dfrac{\kappa}{\mu_i}\nabla\big(p(x) + \gamma_i g x_{d+1}\big) \text{ in } \Omega_i \end{cases}$$

Define ϕ in both phases via $\phi(x) = p + \gamma_i g x_{d+1}$, then

$$\phi, \partial_\nu \phi \text{ continuous across } \Gamma$$

Interfacial Darcy Flows

Example 1: The Muskat Problem for two immiscible fluids

Take $d = 2$. ($d=1$)

If a solution is such that for some time interval we have

$$\Gamma = \{(x, y) \in \mathbb{R}^3 : y = u(x, t)\}$$

then we know (Gancedo and Córdoba, 2007) that $u(x, t)$ solves

$$\partial_t u = c \int_{\mathbb{R}^2} \frac{(\nabla u(x) - \nabla u(x-y)) \cdot y}{(|y|^2 + (u(x) - u(x-y))^2)^{\frac{3}{2}}}\, dy$$

This representation clarifies the parabolic nature of the system.

Example 2.1: The Stefan problem

Let $\varepsilon_0 > 0$, we consider the problem

$$\begin{aligned} \varepsilon_0 \partial_t \phi &= \Delta \phi \text{ in } \{\phi > 0\} \cup \{\phi < 0\} \\ V &= [\partial_\nu \phi] \text{ on } \Gamma = \partial\{\phi > 0\}. \end{aligned}$$

where $[\partial_\nu \phi] = \partial_\nu^+ \phi - \partial_\nu^- \phi$, the jump in the normal derivative.

This is a very different free boundary condition since $\partial_\nu \phi$ will generally be discontinuous across Γ.

Interfacial Darcy Flows

Example 2.2: The (quasistatic) Stefan problem ($\varepsilon_0 \to 0$)

$$\Delta\phi = 0 \text{ in } \{\phi > 0\} \cup \{\phi < 0\}$$
$$V = [\partial_\nu \phi] \text{ on } \Gamma = \partial\{\phi > 0\}.$$

This is the same model as earlier in the talk and the main example we will have in mind.

Interfacial Darcy Flows

Example 3: One phase Hele-Shaw

Saffman and Taylor (ca. 1958): in the Hele-Shaw cell assume

- gravity is negligible
- one of the fluids has negligible viscosity

Then in the remaining phase we have

$$\Delta\phi = 0 \text{ in } \Omega$$
$$\phi = 0 \text{ in } \Gamma$$
$$V = \partial_\nu \phi \text{ on } \Gamma$$

Interfacial Darcy Flows

Example 3: One phase Hele-Shaw

This flow appears in too many places to list here properly!

Internal DLA
RMT
Droplet dynam....

Interfacial Darcy Flows

Example 3: One phase Hele-Shaw

This flow appears in too many places to list here properly!

Here is e.g. one more such instance:
Consider the Porous Medium Equation for $m >> 1$

$$\partial_t p_m = (m-1)p\Delta p_m + |\nabla p_m|^2.$$

As $m \to \infty$, p_m converges to a solution of one phase Hele-Shaw

This limit arises (with some additional terms) in mechanical models of tumor growth (Perthame, Vázquez, Quiros, 2014)

Work with Kim, Mellet (2020)

Interfacial Darcy Flows

Example 3: One phase Hele-Shaw

The theory for the one-phase Hele-Shaw problem is significantly more developed, both theories of solutions as well as regularity

For a small (and highly biased) sample:

Persistence of Lipschitz regularity (King, Lacey, Vázquez 1995)

Phase field limit (Chen and Caginalp 1998, among others!)

Viscosity solutions à la Caffarelli-Vázquez (Kim, 2003)

Flatness implies smoothness (Kim, Choi, and Jerison 2007)

Interfacial Darcy Flows

Example 4: Prandtl-Batchelor flow

This vortex path model leads to the equilibrium problem

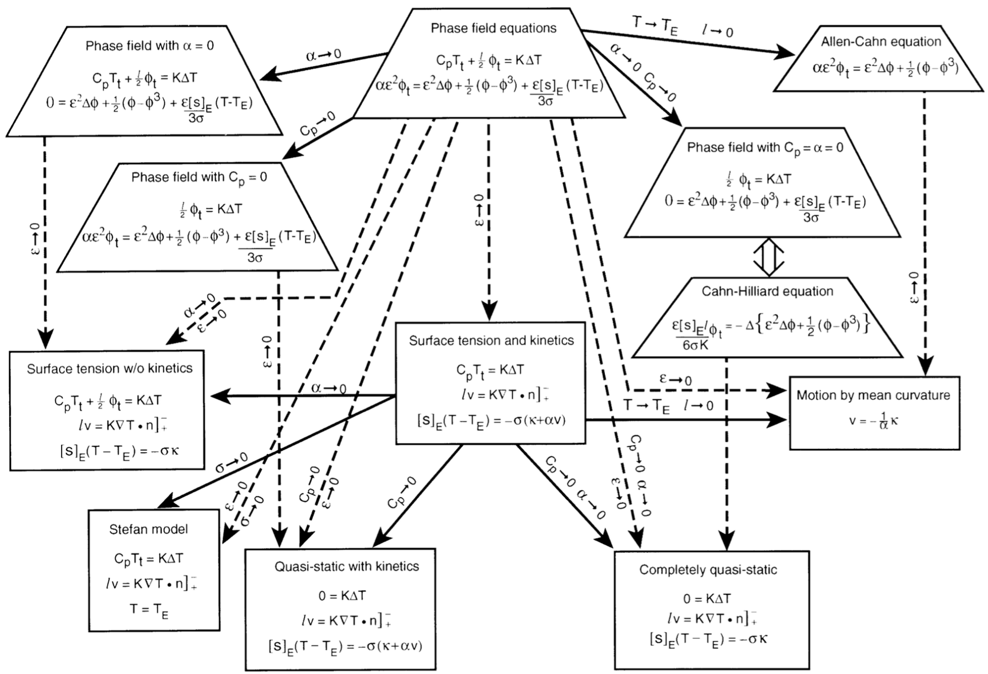
$$\Delta\phi = 0 \text{ in } \{\phi > 0\}$$
$$\Delta\phi = 1 \text{ in } \{\phi < 0\}$$
$$V = 0 = G(\partial_\nu^+\phi, \partial_\nu^-\phi) \text{ on } \Gamma = \partial\{\phi > 0\}.$$

where $G(a,b) = a^2 - b^2 - 1$.

The resulting HJB equation is naturally posed on the sphere, the corresponding theory was developed by Reshma Menon in her doctoral dissertation (2020).

Interfacial Darcy Flows

A map of asymptotic limits (Chen and Caginalp, 1998)



3.The free boundary operator

The free boundary operator

All of these equations can be posed, at least for some time, as

$$\partial_t u = I(u)$$

In essentially all the examples the resulting equation in closely connected to the fractional heat equation $\partial_t u + (-\Delta)^{\frac{1}{2}} u = 0$, and this in turn led to the development of several well posedness theories.

The free boundary operator

For the rest of this talk we focus on the original free boundary problem, henceforth denoted FBP:

$$\begin{cases} \Delta\phi = 0 \text{ in } \{\phi > 0\} \\ \Delta\phi = 0 \text{ in } \{\phi < 0\} \\ V = \partial_\nu^+\phi - \partial_\nu^-\phi \text{ on } \Gamma = \partial\{\phi > 0\} \end{cases}$$

which we recalled was posed on the strip $\mathbb{R}^d \times [0, L]$.

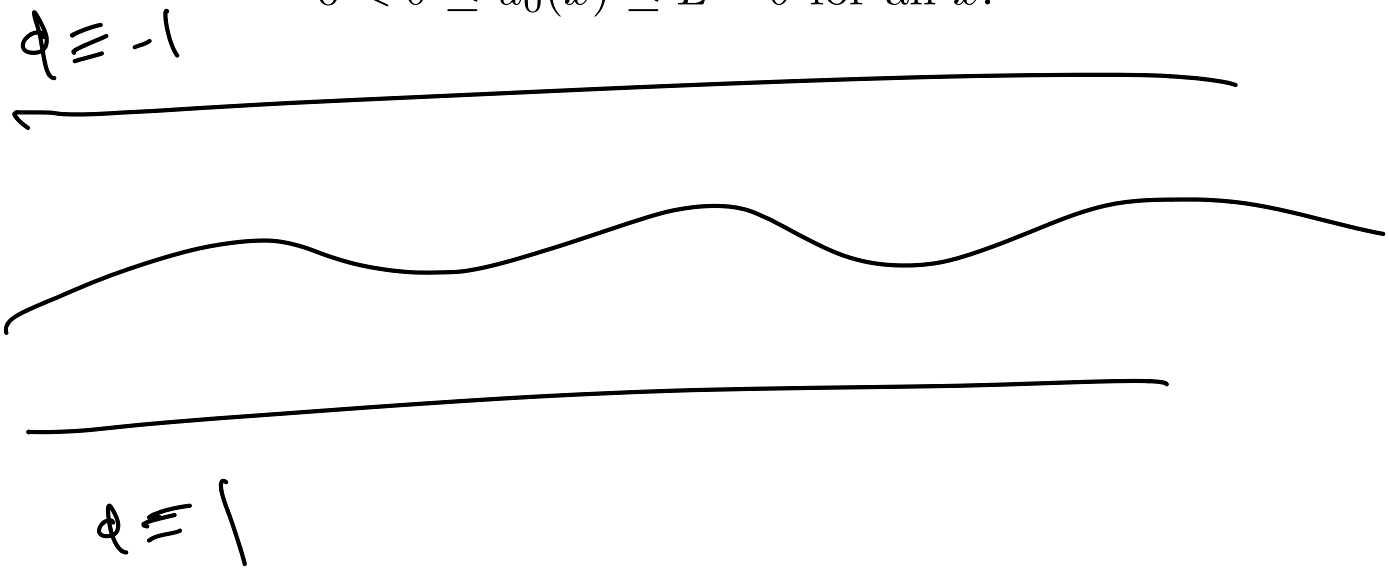
The free boundary operator

Recall also the FBP is posed in the horizontal strip $\{0 \leq y \leq L\}$

$$\phi \equiv 1 \text{ on } \{y = 0\} \text{ and } \phi \equiv -1 \text{ and } \{y = L\}.$$

The initial interface is given by a continuous u_0 such that

$$0 < \delta \leq u_0(x) \leq L - \delta \text{ for all } x.$$

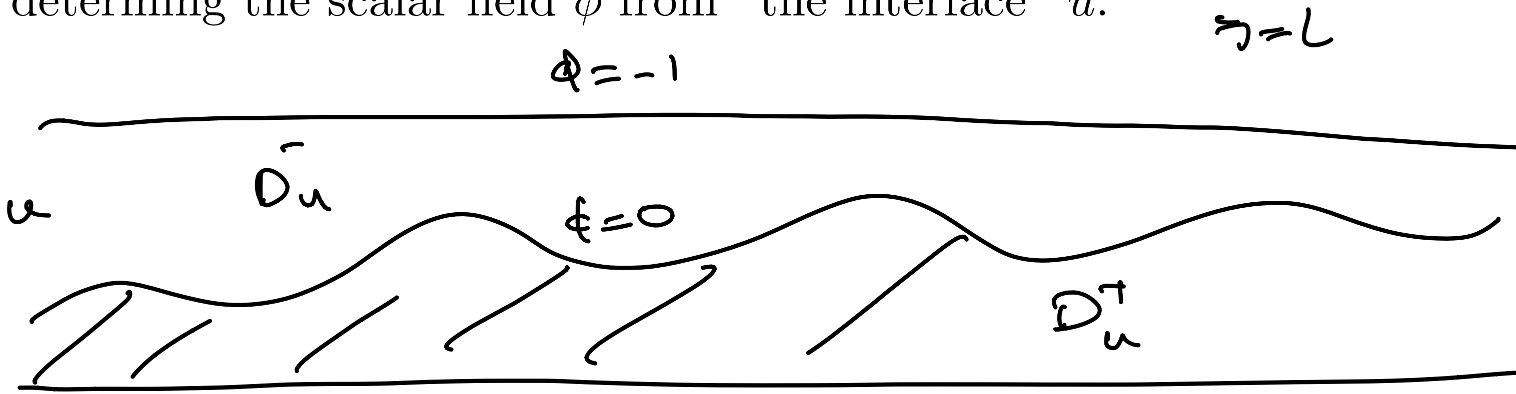


The free boundary operator

Consider the mapping

$$u \mapsto \phi$$

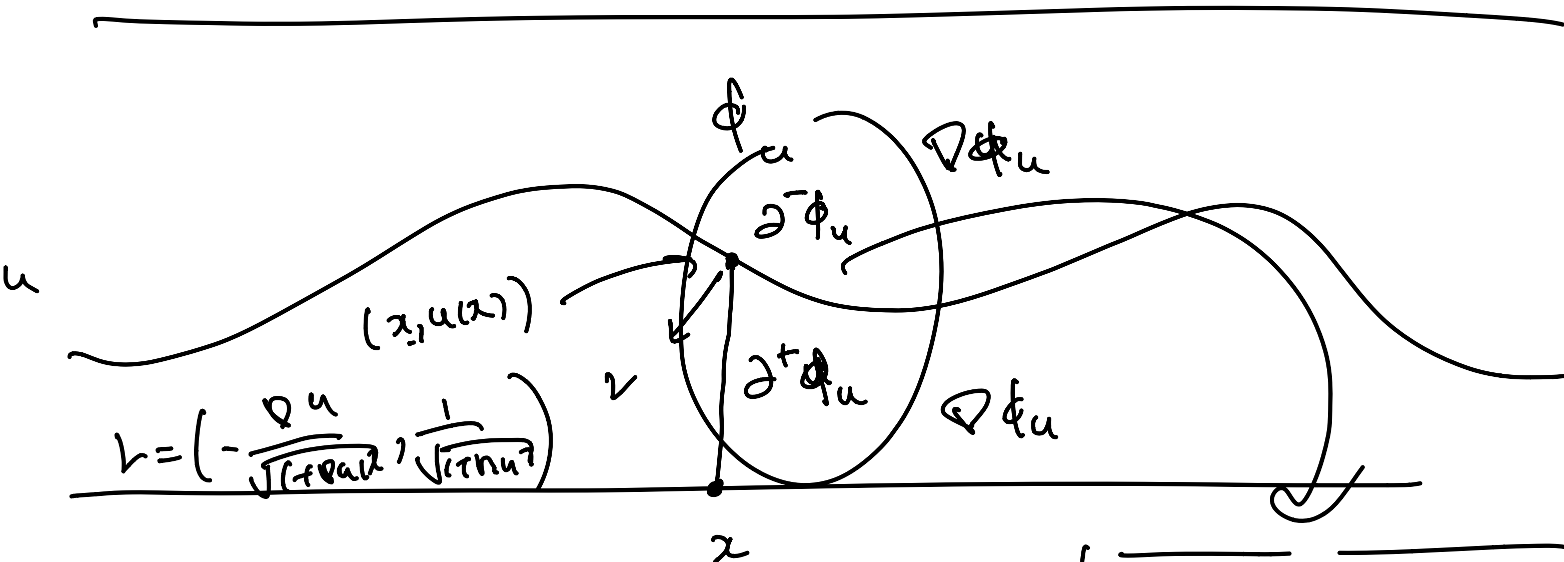
determing the scalar field ϕ from "the interface" u.



Define the sets

$$D_u^+ = \{(x, y) \in \mathbb{R}^{d+1} \mid 0 < y < u(x)\}$$
$$D_u^- = \{(x, y) \in \mathbb{R}^{d+1} \mid u(x) < y < L\}$$

The free boundary operator

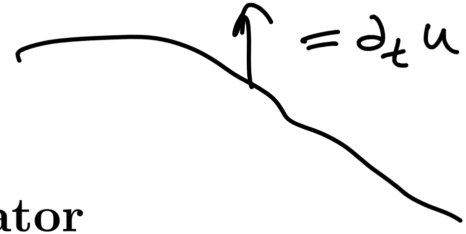


Given u, we let ϕ_u be the **unique** solution to

$$\left\{\begin{array}{l} \Delta\phi = 0 \text{ in } D_u^+ \cup D_u^- \\ \phi = 1 \text{ in } \{y = 0\} \\ \phi = 0 \text{ in } \{y = u(x)\} =: \Gamma_u \\ \phi = -1 \text{ in } \{y = L\}. \end{array}\right.$$

$$\boxed{\partial_\nu^+ \phi_u(x, u(x)) - \partial_\nu^- \phi_u(x, u(x))}$$

The free boundary operator



Then, let us define the **free boundary operator**

$$I(u,x) := \frac{G(\partial_\nu^+ \phi_u, \partial_\nu^- \phi_u)}{\sqrt{1+|\nabla u(x)|^2}}$$

$\partial_\nu^+ \phi_u - \partial_\nu^- \phi_u$

where $\partial_\nu^\pm \phi$ is evaluated at $(x, u(x))$.
The quantity $I(u,x)$ is simply the vertical component of the interface velocity, meaning that

$$\partial_t u = I(u,x)$$

The free boundary operator

Solving the FBP amounts to solving the Cauchy problem

$$\begin{cases} \partial_t u &= I(u,x) \text{ in } \mathbb{R}^d \times (0,\infty) \\ u &= u_0 \qquad\quad \text{at } t=0 \end{cases}$$

Now, we recall the theorem stated at the beginning.

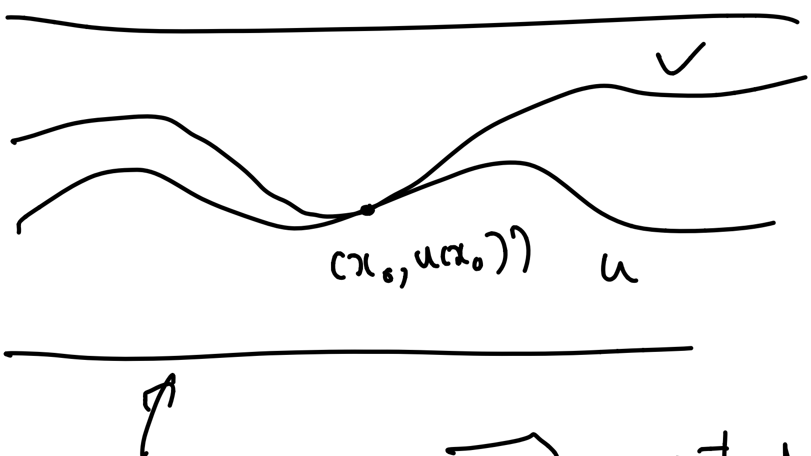
Theorem (2019, Nonlinear Analysis)

There is a unique weak solution $u(x,t)$ to the Cauchy problem and the comparison principle holds. In particular, any spatial modulus of continuity of u is propagated forward in time.

The free boundary operator

Proposition
The free boundary operator I has the *Global Comparison Property* (GCP). Namely, if u, v are two smooth functions such that $u \leq v$ in $\mathbb{R}^d$ and $u = v$ at x_0, then $I(u, x_0) \leq I(v, x_0)$.

$\partial_y \phi = 0$



$\phi_v \geq \phi_u$ everywhere

$\Rightarrow \partial^+_\nu \phi_u \leq \partial^+_\nu \phi_v$

$-\partial^-_\nu \phi_u \leq -\partial^-_\nu \phi_v$

$g(u, u_\nu, x) = 0$

$\Rightarrow$

$$\frac{\partial^+_\nu \phi_u - \partial^-_\nu \phi_u}{\sqrt{1+|\nabla u|^2}} \leq \frac{\partial^+_\nu \phi_v - \partial^-_\nu \phi_v}{\sqrt{1+|\nabla v|^2}}$$

$$Iu(x_0) \leq I(v)(x_0)$$

The free boundary operator

$\partial_t u = (1 - |\nabla|)\, \Delta u$

$F\big(u(x+h) - u(x),\ \nabla u(x),\ h\big)$

Another example

In the Muskat problem, one has the alternative expression

$$I(u,x) = c \int_{\mathbb{R}^2} \frac{u(x+h) - u(x) - \nabla u(x) \cdot h}{(|h|^2 + (u(x+h) - u(x))^2)^{\frac{3}{2}}}\, dh$$

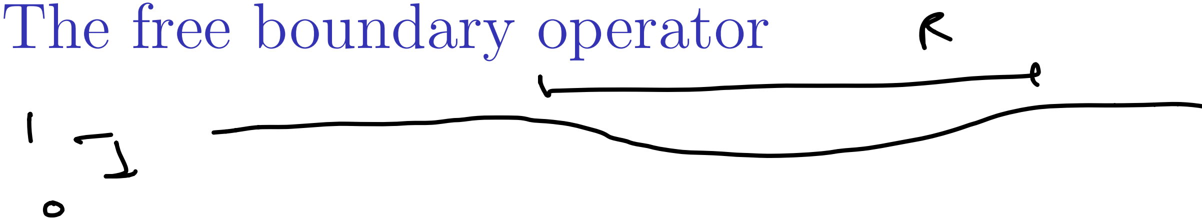
5

Let u and v be two functions with Lipschitz norm ≤ 1.
Suppose v touches u from above at x_0, then

$$I(u, x_0) \leq I(v, x_0).$$

From here follows the propagatation of Lipschitz norms ≤ 1, from where higher regularity follows (see work of S. Cameron).

The free boundary operator



It remains to show the second property for I, namely:

Let u lie in some fixed compact set.
Then, given $\varepsilon > 0$ there is $R > 0$ such that

$$I(u + C + h\phi_R, x) < I(u,x) + \varepsilon h, \ \forall\, C, h > 0$$

Here, $\phi_R(x) = \phi(x/R)$ and $\phi(x) = |x|^2/(1+|x|^2)$

The free boundary operator

Proving this is not as straightforward as the first property!

Let us show it for $I = \Delta^{\frac{\alpha}{2}}$. By linearity, this reduces to:

Given $\varepsilon > 0$ there is $R > 0$ such that

$$\phi(x+h) - \phi(x) - 1_{B_1}(h) \cdot \nabla\phi(x) \cdot h \qquad \Delta^{\frac{\alpha}{2}} \phi_R \leq \varepsilon. \qquad \forall x, h$$

This in turn follows from $|\delta_h \phi_R(x)| \leq CR^{-2}|h|^2$ for all $h \in \mathbb{R}^d$.

$$\Delta^{\frac{\alpha}{2}}\phi = c \int \underbrace{\delta_h \phi(x)}_{\boxed{(\phi(x+h) - \phi(x) - 1_{B_1}(h) \cdot \nabla u(x) \cdot h)}} |h|^{-d-\alpha}\, dh \approx o(1) \quad R \to \infty$$

4.The GCP and Lévy operators

The GCP and Lévy operators

In the 1960's, Courrège considered linear operators

$$L : C_b^2(\mathbb{R}^d) \to C_b^0(\mathbb{R}^d)$$

and showed that if L has the GCP then it has the form

$$\underbrace{c(x)f(x) + \boldsymbol{b}(x)\cdot\nabla f(x) + \mathrm{tr}(\mathrm{A}(x)D^2 f(x))}_{\text{drift-react-diff (local)}} + \underbrace{\int_{\mathbb{R}^d} \delta_h f(x)\,\nu(x,dh)}_{\text{jump part.}}$$

The GCP and Lévy operators

Here, for the sake of concise notation, we are writing

$$\int_{\mathbb{R}^d} \delta_h f(x)\, \nu(x, dh)$$

where

$$\delta_h f(x) := f(x+h) - f(x) - \chi_{B_1}(h) \nabla f(x) \cdot h$$

For each x, $\nu(x, dh)$ is a Lévy measure, meaning that

$$\int_{\mathbb{R}^d} \min\{1, |h|^2\} \nu(x, dh) < \infty$$

The GCP and Lévy operators

Euclidean

In a previous work with Schwab (2019), we extended Courrège's result to nonlinear operators

Bellman
Isaacs
operator

$$I(u,x) = \min_{\alpha} \max_{\beta} \{ f_{\alpha\beta} + L_{\alpha\beta}(f,x) \}$$

For our problem

where, for every α and β we have

$$L_{\alpha\beta}(f,x) = c_{\alpha\beta} f(x) + \boldsymbol{b}_{\alpha\beta} \cdot \nabla f(x) + \int_{\mathbb{R}^d} \delta_h f(x)\, \nu_{\alpha\beta}(dh)$$

$|h|^{1+\varepsilon}$

The GCP and Lévy operators

Using the min-max representation, the second property follows relatively easily, since

$$I(u + C + h\phi_R, x) \leq I(u, x) + \sup_{\alpha\beta} L_{\alpha\beta}(C + h\phi_R, x)$$

and all of the terms $L_{\alpha\beta}(\cdot)$ can be estimated as done with the fractional Laplacian

$$\int_{\mathbb{R}^d} \overbrace{\delta_h \phi_R}^{\lesssim |h|^2/R^2} \, \nu_{\alpha\beta}(dh)$$

5.Regularity questions

Regularity questions

Problem
Show if $\underset{u_0}{\cancel{f_0}}$ is Lipschitz, then $\overset{u}{\cancel{f}}(x,t)$ is smooth for every $t > 0$.

If u is Lipschitz then the min max
formula for I can be improved

$$I(u,x) = \min_{\alpha} \max_{\beta} \left\{ c_{\alpha\beta} + \underbrace{L_{\alpha\beta}(u,x)}_{\int_{\mathbb{R}^n} \delta_h u \, \nu_{\alpha\beta}(dh)} \right\}$$

$\approx K(h)\,dh$?

$\approx |h|^{-d-1}\,dh$?

Regularity questions

see our Cam

This point is illustrative of an important difference between the Muskat problem and our problem

$$\frac{1}{\left(|x-y|^2 + (u(x)-u(y))^2\right)^n}$$

Muskat:
Lipschitz: more difficult! Lipschitz⇒smoothness: easier!

Two phase QS Stefan:
Lipschitz: easier! Lipschitz⇒smoothness: more difficult!

no good form for I.

Regularity questions

For the FBP, Abedin and Schwab (2020) proved the following:

a

If $\cancel{f}(x,t)$ has a spatial gradient which is Dini continuous for every t, then $\cancel{f}$ is $C^{1,\alpha}$

u

Limitations of the framework and future work

Limitations of the framework and future work

Equations with variable coefficients, well-posedness?
(Potentially useful for studying problems in heterogeneous media)

What happens if f is Lipschitz?
(This requires understanding the Lévy measures arising in the min-max representation)

What about the Stefan problem?
(One could develop a similar theory, but now you are dealing with nonlocal space-time operators)

Limitations of the framework and future work

Far more substantial limitations are:

Method disregards important divergence/variational structure

Handling surface tension (a nonlocal 3rd order equation)

Data at low regularity: what happens to singularities?

Thank you!

Comments / Questions / Suggestions

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